Review of Algebra

Arithmetic Operations

$a + b = b + a$	$ab = ba$	Commutative Law
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$	Associative Law
$a(b + c) = ab + bc$		Distributive Law

Examples: (a) $(3xy)(-4x) =$

(b) $1 + 4x^2 - 3x(x - 2) =$

Applying the Distributive Law three times gives

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$$

Each term in one factor multiplies each term in the other factor and the products are added. Some common special cases are:

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (a - b)^2 = a^2 - 2ab + b^2$$

Examples: (a) $(2x + 1)(3x - 5) =$

(b) $(x + 5)^2 =$

(c) $(x + 3)(x - 2)(4x + 1) =$

Fractions

The *inverse* of a number a is the number, denoted a^{-1}, such that $a \cdot a^{-1} = 1$.
A *fraction* $\dfrac{a}{b}$ is just another way to write $a \cdot b^{-1}$: $a \cdot b^{-1} = \dfrac{a}{b}$.
In particular,

$$\frac{1}{a} = a^{-1} \qquad \frac{a}{a} = a \cdot a^{-1} = 1$$

Since $(ab)(a^{-1}b^{-1}) = (a \cdot a^{-1})(b \cdot b^{-1}) = 1$, we see that $(ab)^{-1} = a^{-1}b^{-1}$.

To multiply two fractions, just multiply the numerators and the denominators:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = (a \cdot b^{-1})(c \cdot d^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}}$$

Note that $\dfrac{-a}{b} = -\dfrac{a}{b} = \dfrac{a}{-b}$.
To add two fractions with the same denominator, we use the Distributive Law:

$$\boxed{\frac{a}{b} + \frac{c}{b} = a \cdot b^{-1} + c \cdot b^{-1} = (a+c)b^{-1} = \frac{a+c}{b}}$$

Remember: $\dfrac{a}{b+c} \neq \dfrac{a}{b} + \dfrac{a}{c}$.
To add two fractions with different denominators, first find a common denominator:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}}$$

This process is often called *cross-multiplication*.
To divide two fractions:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{ad}{bc}}{\frac{cd}{cd}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$$

This amounts to inverting the denominator and multiplying:

$$\boxed{\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}}$$

Examples: (a) $\dfrac{x+3}{x}$

(b) $\dfrac{3}{x-1} + \dfrac{x}{x+2} =$

(c) $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$

Factoring

Reversing the process of the Distributive Law is called *factoring*.
For example, $3x^2 - 6x = 3x(x - 2)$.
To factor a quadratic of the form $x^2 + bx + c$, we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

so we need to find numbers r and s, whose sum $r + s = b$ and whose product $rs = c$.

Examples: (a) Factor $x^2 + 3x + 2$.

(b) Factor $x^2 + 5x - 24$.

Some common expression can be factored easily:

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ a^3 + b^3 &= (a+b)(a^2-ab+b^2) \end{aligned}$$

Examples: Factor the following polynomials:
(a) $x^2 - 10x + 25$

(b) $9x^2 - 16$

(c) $x^3 + 8$

(d) Simplify $\dfrac{x^2 - 9}{x^2 + x - 12}$

To factor higher degree polynomials, it is useful to remember the following fact about a polynomial $p(x)$:

If $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Example: To factor $p(x) = x^3 - 13x + 12$ we first note that $p(1) = (1)^3 - 13(1) + 12 = 0$. Therefore $p(x) = (x - 1)q(x)$. To find $q(x)$ we use *long division*.

Completing the Square

In order to graph $\boxed{y = ax^2 + bx + c}$ or solve for its roots, the technique of *completing the square* is very useful. The idea is to **rewrite** y **as** $\boxed{y = a(x+p)^2 + q}$:

$$ax^2 + bx + c = a(x+p)^2 + q = a(x^2 + 2px + p^2) + q = ax^2 + 2apx + ap^2 + q$$

By equating the coefficients, we find that

$$b = 2ap, \qquad c = ap^2 + q$$

Solving these equations gives $p = \dfrac{b}{2a}$ and $q = c - ap^2 = c - a\dfrac{b^2}{4a^2} = c - \dfrac{b^2}{4a}$ so that

$$ax^2 + bx + c = a\Big(x + \frac{b}{2a}\Big)^2 + \Big(c - \frac{b^2}{4a}\Big)$$

These formulas need not be memorized. The steps taken above are simple enough to carry out for any given example. We may, however, derive the *Quadratic Formula* from this expression. To solve $ax^2 + bx + c = 0$ we isolate the square term in the above expression:

$$a\Big(x + \frac{b}{2a}\Big)^2 + \Big(c - \frac{b^2}{4a}\Big) = 0 \Longleftrightarrow a\Big(x + \frac{b}{2a}\Big)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a} \Longleftrightarrow \Big(x + \frac{b}{2a}\Big)^2 = \frac{b^2 - 4ac}{4a^2}$$

Then we take the square root to obtain the final formula for the roots:

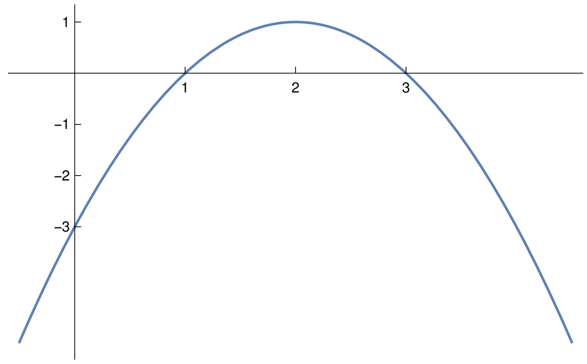
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Longleftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Fomula. The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example: Let $y = -x^2 + 4x - 3$. To complete the square, we solve for p and q by equating coefficients:

$$-x^2 + 4x - 3 = -(x+p)^2 + q = -x^2 - 2px - p^2 + q$$

The graph of an equation of the form $y = -(x+p)^2 + q$ is that of an inverted parabola (opening down) with vertex at the point $(-p, q)$.
Sketch the graph of the above parabola and include the points where it crosses the x-axis:



1
1
2
3
−1
−2
−3

Binomial Expansion

When expanding binomial expressions of the form $(a+b)^n$ a pattern emerges.

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

The exponents on a start at n and decrease to 0, while the exponents on b start at 0 and increase to n. The coefficients follow a pattern called *Pascal's Triangle*.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The number on any line is the sum of the two numbers above it on the previous line.

Examples: (a) $(x-2)^4 =$.

(b) $(3x+1)^5 =$

(a) $(x+h)^3 =$.

Radicals

The symbol $\sqrt{\ }$ means "the positive square root of." Thus,

$$x = \sqrt{a} \qquad \text{means} \qquad x^2 = a \quad \text{and} \quad x \geq 0$$

Note that $a \geq 0$ since it equals a square of a number which is always non-negative.

Square roots work well with products and quotients,

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

but *not* with sums or differences,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \qquad \sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$$

For example $\sqrt{9+16} = \sqrt{25} = 5$, not $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

Examples: (a) Sinplify $\dfrac{\sqrt{50}}{\sqrt{2}}$

(b) $\sqrt{x^2y} =$.

Note that $\sqrt{x^2} = |x|$ because $\sqrt{\ }$ indicates the *positive* square root.

Examples: (a) Sinplify $\sqrt{(-10)^2}$

(b) If $x < 0$, is $x = \sqrt{x^2}$?

(c) Sinplify $\sqrt{x^3}$

In general,

if n is a positive integer, $x = \sqrt[n]{a}$ means $x^n = a$. If n is even then $a \geq 0$ and $x \geq 0$.

The same rules hold for these more general roots.

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Examples: (a) $\sqrt[3]{-64} =$.

(b) $\sqrt[5]{x^6} =$.

(c) If $x < 0$, is $x = \sqrt[3]{x^3}$?

To "*rationalize*" a numerator or denominator that contains an expression such as $\sqrt{a}-\sqrt{b}$, we multiply the the numerator and denominator by the "*conjugate*" radical $\sqrt{a}+\sqrt{b}$. Then we can take advantage of the formula for the difference of two squares

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$

Example: $\dfrac{\sqrt{x+5}-3}{x-4} =$

Exponents

For any positive integer n, a^n is shorthand for multiplying a by itself n times. By convention, we let $a^0 = 1$ and $a^{-n} = \dfrac{1}{a^n}$ so that exponents are defined for all integers. We define fractional exponents by the rules $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = (\sqrt[n]{a})^m$. With these conventions, the following rules are always valid.
Laws of Exponents. For any rational numbers r and s

$$a^r \cdot a^s = a^{r+s} \qquad \frac{a^r}{a^s} = a^{r-s} \qquad (a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$$

Notice that there are no similar rules involving addition or subtraction: $(a+b)^r \neq a^r + b^r$.

Examples: (a) $3^7 \times 27^4$

(b) $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$

(c) $4^{3/2} =$.

(d) $\dfrac{1}{\sqrt[5]{x^3}} =$

(e) $\left(\dfrac{x}{y}\right)^{-2} \left(\dfrac{y^2 x}{z}\right)^3$

Miscellaneous

When adding two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, we can create the common denominator bd and add to get

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

If b and d have common factors, we can sometimes benefit from finding a "smaller" common denominator k for which both b and d divide k.

Examples: (a) $\frac{(x-1)}{x^2-4}+\frac{(x+1)}{(x-2)(x+3)}$

(b) $\frac{(x-2)}{(x-1)^2(x-3)}+\frac{(x+1)}{(x-1)(x+3)}$

(c) $\frac{5}{6}+\frac{3}{10}$

When calculating limits, we often need to write examples such as those shown below as a single root:

Examples: (a) For $x > 0$, $\dfrac{\sqrt{x^2+x-1}}{x} =$

(b) For $x < 0$, $\dfrac{\sqrt{x^2+x-1}}{x} =$

(c) For $x < 0$, $\dfrac{\sqrt[3]{x^3+x^2-1}}{x} =$